Trade-Offs in Fair Redistricting

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ABSTRACT

What constitutes a 'fair' electoral districting plan is a discussion dating back to the founding of the United States and, in light of several recent court cases, mathematical developments, and the approaching 2020 U.S. Census, is still a fiercely debated topic today. In light of the growing desire and ability to use algorithmic tools in drawing these districts, we discuss two prototypical formulations of fairness in this domain: drawing the districts by a neutral procedure or drawing them to intentionally induce an equitable electoral outcome. We then generate a large sample of districting plans for North Carolina and Pennsylvania and consider empirically how *compactness* and *partisan symmetry*, as instantiations of these frameworks, trade off with each other – prioritizing the value of one of these necessarily comes at a cost in the other.

CCS CONCEPTS

• **Applied computing → Law, social and behavioral sciences**; *Voting / election technologies*; • **Theory of computation** → *Random walks and Markov chains.*

KEYWORDS

redistricting, gerrymandering, fairness, compactness, partisan symmetry, Pareto-optimal, Markov chain Monte Carlo, Pareto frontier

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1 INTRODUCTION

Gerrymandering, the careful crafting of electoral districts to favor or disfavor a particular outcome, is a hot topic in contemporary political discourse. In advance of the 2020 U.S. Census and subsequent redistricting processes, several high-profile court cases, reform initiatives, and new lines of academic research have ignited discussions about what kinds of processes and outcomes lead to the 'fairest' districts. However, fairness in this setting is loosely defined. Since the early days of the republic, politicians have used the power of the pen to draw districts which help their political allies and harm their rivals. The term *gerrymander* itself comes from a portmanteau used in an 1812 political cartoon lampooning Massachusetts governor Elbridge *Gerry*[1] and a sala*mander*-shaped state senate district which was part of a plan advantaging the governor's Democratic-Republican party. Since then, districts and districting plans have been identified as unfair for various reasons, but a singular framework for determining when a districting plan is *fair* remains elusive.

Since the early 1960s, advocates for fair districts and districting procedures have proposed using algorithmic techniques to remove the human element, and therefore potential for human bias, from the system. In a letter, economist William Vickrey proposed an algorithmic framework with a large amount of randomness to even further separate human decisions from the eventual output [29]. Over the last sixty years, the growth in computational power and availability of data brings us to a point where Vickrey's dream of an autonomous redistricting machine could be realized [2]. However, the use of an algorithm does not imply that the internal process of drawing the lines or the districting plan it outputs is unbiased or fair. Given the renewed interest in the redistricting problem, the emergence of computational districting methods in legal settings, and the availability of the necessary resources to properly implement an algorithmic redistricter, it is important to understand how differing views of fairness may or may not be compatible with each other in such a system.

Our Work. We begin by highlighting some recent algorithmic approaches to drawing districts. We then discuss two conceptualizations of fairness in this domain: drawing districts by a *neutral process* and drawing districts to achieve a particular *outcome* which aligns with certain values. We consider these two approaches in an empirical domain using computer-generated districting plans for North Carolina and Pennsylvania and construct Pareto frontiers to examine the trade-off between optimizing for the *compactness* of the districts and the *partisan symmetry* of the contests in those districts. Finally, we discuss some future directions for inquiry and research in the domain of automated and algorithmic redistricting.

1.1 Automated Redistricting

Several works have proposed purely algorithmic approaches to constructing electoral districts, and the prototypical formulation is to minimize a functional evaluating a geometric property of the districting plan, subject to a few standard constraints including population balance and connectedness. Vickrey's proposal as well as the algorithms of Levin and Friedler [20], Chen and Rodden [8, 9], Hess et al. [17], and Cohen-Addad et al. [11] involve selecting a random location as a 'seed' for each district and then assigning territory to each of those seeds based on proximity in a particular

[1] Pronounced with a hard 'g' as in *grant*.

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way, such as with Voronoi diagrams or an iterative flood fill procedure. The shortest splitline algorithm [18, 26] and the diminishing halves algorithm [27] choose to iteratively cut the state along the shortest line meeting a particular criterion. For a more detailed overview of these algorithms, see the introduction of [20]. Other computational redistricting techniques include the Markov chain Monte Carlo approach [3, 7, 10, 12, 13] and simulated annealing [6], which involve making random perturbations to a districting plan to improve its score according to some measure, as well as genetic algorithms [21]. Both of these can be used to search for a maximally compact plan as the other algorithmic approaches do, but can also be instantiated with other objective functions, and so have a more versatile functionality but are less clear in how they arrive at a particular 'final' plan.

2 PROCEDURAL NEUTRALITY

The first conceptualization of fairness we discuss is that of a *neutral process*; districts ought to be drawn without considering of any of the potentially sensitive attributes of the underlying population such as racial or partisan information. The most common proposal under this framework is to draw districts which maximize a particular notion of *compactness* subject to the basic constraints of connectedness and equal population; indeed this is the underlying objective of several of the algorithms outlined in the previous section. From the legal side, many jurisdictions specify that districts should split political subunits, such as counties or municipalities, as little as possible. The degree to which the preservation of political subunits binds the process in practice varies widely. It is treated very seriously, for example, in Iowa and West Virginia where the congressional districts enacted after the 2010 Census do not split any counties. In other states, it is treated more as a guiding principle.

Taking neutrality as a definition of fairness has several advantages. First, many of the *redistricting principles* [1, 25], including contiguity, compactness, avoidance of partisan data, and preserving political subunits, fall under the framework of neutrality. Additionally, these neutral criteria are typically easy to operationalize and quantify. For example, we can take simply count the number of municipalities two districting plans split and objectively observe that one splits fewer than the other. Such an analysis is not as straightforward for the outcome-centered criteria described in the next section. Stern [28] argues that by rigorously adhering to a standard of compactness, districts may contain fragments of many different communities, encouraging the formation of coalitions, which then has a positive impact on the democratic process. For these reasons, a clear set of neutral criteria for drawing districts has been a common approach for redistricting reform since the 1960s [15].

On the other hand, districts being composed of fragmented communities can impede the ability of minority groups to achieve representation in the legislative body. The 1982 amendments to the Voting Rights Act and subsequent court opinions specifically prohibit this kind of fragmentation, colloquially referred to as the 'cracking' of voters. Through the history of the United States, political mechanisms including the drawing of district lines have been used to intentionally limit the political power and access of minority groups. Arguments along these lines admit that adhering solely to neutral criteria can perpetuate these kinds of inequities, and this undermines the use of neutrality as the standard of fairness in this context.

3 FAIRNESS OF OUTCOME

At the other end of the spectrum is that we should only consider the outcomes of the elections in the districts, irrespective of the procedure used to actually generate those districts. Arguments with respect to this viewpoint underpinned several high-profile court cases in recent years, including cases where Democrats earned over half of the statewide vote but a minority of seats. Roughly half of the vote translated into winning 36 of 99 seats in Wisconsin's General Assembly and three of thirteen of North Carolina's congressional districts, for example. In Maryland, the Democratic legislature redrew the state's congressional districts to tilt the partisan balance in one district so as to force a long-time Republican incumbent to narrowly lose to a Democratic challenger.

If we demand that districts lead to a fair outcome, the question which must be addressed is how to define a fair outcome? Even restricting to a solely partisan perspective, where we simply ask for the outcomes to be fair with respect to the voters' partisan identities, this question is very hard to answer. The idea of *proportionality*, that each party should win a fraction of the districts (roughly) equal to its statewide vote share, seems appealing for its simplicity, but it is often not possible to achieve. In Massachusetts, for example, Republicans typically win approximately 35 percent of the statewide vote share in Senatorial and Presidential elections, and a demand for proportionality would demand they win approximately three of the nine congressional seats in the state. However, because Republican voters are distributed roughly evenly around the state, it is very difficult to draw even a single district with a reliable Republican majority, let alone three districts [16].

A similar issue of geographic concentration appears when designing districts which satisfy a notion of proportionality with respect to providing minority groups the ability to elect a candidate of choice. This is further complicated by the observation that, while electing a Republican candidate requires a majority (or at least a plurality) of voters in a district to favor the Republican, electing a minority group's candidate of choice does not require drawing a district in which that group constitutes a majority if there are other voters who will reliably support that group's favored candidate.

Using outcomes as the baseline for fairness is sensible for many reasons. First, if the purpose of representative government is to represent the populace, then any evaluation of fairness should be with respect to the groups and viewpoints elected from the districts to the legislature rather than the process by which the districts themselves come about. Providing communities-of-interest and historically marginalized groups access to representation requires drawing the districts in a way that facilitates these desired outcomes because a neutral process risks fragmenting these communities. Additionally, there are other redistricting principles which require considering the outcomes of potential elections, such as avoiding pitting two incumbents against one another. On the other hand, many seemingly desirable 'fair' outcomes are mutually exclusive. Even narrowly focusing on partisan measures, one person may

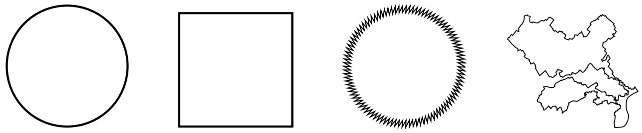

Figure 1: Polsby-Popper scores of four example regions: a perfect circle, a square, a circle with a ragged boundary, and a district from the Pennsylvania plan shown in Figure 11.

believe that districts ought to facilitate as near a proportional outcome as is possible while another may believe that they should be drawn such that the individual district-level elections are as competitive as possible. These are, of course, largely incompatible ideas, since if the elections are competitive, then a small surge in support for one party will tip several of the seats, resulting in a highly disproportionate outcome.

4 EMPIRICAL EVALUATION

To empirically evaluate a quantitative trade-off between adhering to different conceptualizations of fairness, we need to first pin down a metric by which to evaluate a districting plan along each of these dimensions. Here, we select two measures from the literature and use them to evaluate a computer-generated sample of districting plans. We demonstrate that this setting has a clear trade-off between using the two different notions of fairness described previously.

For procedural neutrality, we take the maxim that 'districts ought to be drawn to be as compact as possible'. We measure the compactness of the districts with the *Polsby-Popper compactness score*, which is the most common such measure in the literature and discourse. The score of a region Ω is computed as

$$\mathrm{PP}(\Omega) = \frac{4\pi \times \mathrm{Area}(\Omega)}{\mathrm{Perim}(\Omega)^2}.$$

The Polsby-Popper score measures the normalized ratio of a district's area to the square of its perimeter and takes the form of an an *isoperimetric quotient*. With respect to this measure, a circle is the most compact shape with a Polsby-Popper score of one, and deviations from this ideal decrease the score towards zero. This score is not without its flaws, in particular it is highly sensitive to minor perturbations of the boundary, which may penalize features like coastlines in an undesireable way. We compute a few basic examples of this score in Figure 1; for a modern treatment of isoperimetry in this context, see [14] and for some of the issues with measuring districts' compactness, see [4, 5]. We take the Polsby-Popper score of a districting plan to be the simple arithmetic mean of the scores of its constituent districts.

As our measure of fairness-of-outcome, we use a measure of *partisan symmetry*, evaluating the extent to which Democratic and Republican voters are treated equally under a districting plan. To make this more concrete, we briefly introduce the *seats-votes curve*, which uses the results of an election to extrapolate the necessary statewide vote share for a party to win a particular number of seats. We describe a simple example here, illustrated in Figure 2. Suppose in our fictional election, the Republican party earned 45 percent of the statewide vote share across five individual district contests. In

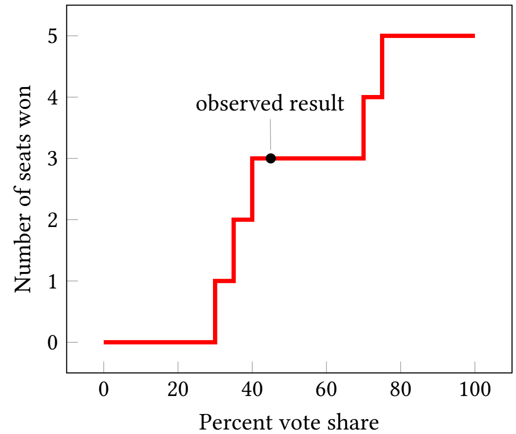

Figure 2: An example of a seats-votes curve.

these races, they won 20 percent, 25 percent, 55 percent, 60 percent, and 65 percent of the vote, respectively, and therefore winning three of the five seats. The point (45,3) is therefore on the seats-votes curve for this election. We can also see that if the Republican vote share increased or decreased a little bit, the number of resulting seats would not change, so points such as (48,3) and (42,3) are also on our seats-votes curve. However, if the Republicans' statewide vote share dropped by seven percent or increased by 27 percent, the number of seats they win would change, so points like (38,2) and (72,4) are also on the seats-votes curve. Performing this exercise for all potential vote shares yields the final curve. The modelling assumption that the percentage point change in vote share is equal across all districts is called the *uniform partisan swing* assumption and is discussed thoroughly by Katz et al. [19].

The seats-votes curve is a simple but powerful picture which captures many standard notions of partisan asymmetry including the *mean-median* score, which measures how far a party's statewide vote share is from its vote share in the median district, and the *efficiency gap* which measures how many votes one party wastes relative to the other. Here, we choose a measure designed to capture asymmetry at all points in the picture: we compute the area between the seats-votes curve as described above and its *inversion* about the midpoint of the figure [24]. This synthesizes, over all vote shares x, how different the number of seats the Democrats would win with x percent of the vote versus the number of seats the Republicans would win with x percent of the vote. In other words, there is an asymmetry if Republicans win y seats with $50 + x$ percent of the vote but do not lose y seats with $50 - x$ percent of the vote. The area between the seats-votes curve and its inversion about the midpoint is the sum over all possible vote shares of the amount of asymmetry for all of the $50 + x$ and $50 - x$ percent pairs. By dividing this area by the total number of seats and subtracting from one, we obtain a score between zero and one, where a score of one means that the plan is perfectly symmetric with respect to both parties, and the score declines towards zero as one party is better able to translate votes to seats, relative to the other.

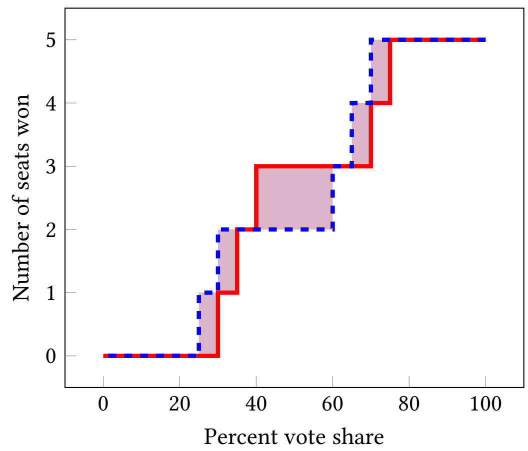

Figure 3: The partisan symmetry score from the example seats-votes curve in Figure 2, drawn with a solid red line and its inversion about the midpoint, $(50, 2.5)$, drawn with a dashed blue line. The shaded area corresponds to the amount of asymmetry, and this plan achieves a score of 0.92.

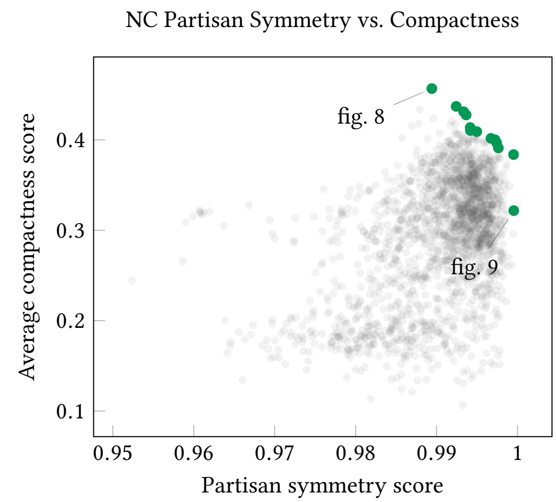

Figure 4: Comparison of partisan symmetry and compactness for North Carolina. Each point corresponds to one plan in the sample. Green points are Pareto-optimal, grey points are Pareto-dominated.

4.1 Generating Plans

We use the GerryChain Python package [22] to examine hypothetical districting plans for two states: North Carolina's 13 congressional districts and Pennsylvania's 18 congressional districts. Both of these states are reasonably close to having an equal number of Democrats and Republicans and both have had high-profile court cases challenging their congressional districts in recent years. The data for both states comes from the mggg-states repository on GitHub [23]. We evaluate the partisan symmetry score using a statewide U.S. senatorial race for both states, the 2014 election in North Carolina and the 2016 election in Pennsylvania. In both contests, the Republican candidate narrowly won the election.

We are interested in finding plans at the *Pareto frontier* of compactness and partisan symmetry; districting plans for which there is no other plan which is both more compact and has a higher degree of partisan symmetry. We call a plan on the Pareto frontier *Pareto-optimal* and one that is not we call *Pareto-dominated*. Because the collection of all districting plans which meet the basic criteria of connectedness and population equality is unfathomably large, directly constructing plans of interest is extremely challenging. Instead, GerryChain allows us to use a Markov chain Monte Carlo procedure to generate a large number of plans and extract the Pareto-optimal subset as an approximation to the true Pareto frontier. In brief, our algorithm first generates a random plan then attempts to make small random modifications which improve either its compactness, its partisan symmetry, or both, thereby performing a guided random walk through the space of districting plans. After repeating this for a large number of random seeds and a large number of steps for each walk, we can extract all of the Pareto-optimal plans and use these to draw the empirical Pareto frontier.

The data is in the form of a graph with a vertex for each *voting tabulation district* (VTD), which is the smallest geographic units at which election results are aggregated. Two vertices are joined by an edge if their corresponding VTDs are geographically adjacent. All modifications to plans are made at this level, that is, our problem can be viewed as a *graph partitioning problem* where each VTD must be assigned to exactly one district. For this reason, the universe of possible plans this procedure can generate is more restricted than when working with smaller units such as U.S. Census blocks or drawing free-hand contours through the state. The only constraints we use are connectedness and population equality, which for the sake of tractability is taken to mean that a districting plan is valid if the deviation from the ideal of the population of any district is no more than 2.5 percent.

4.2 Results

In Figures 4 and 5, we plot the compactness and partisan symmetry of our samples of plans, highlighting the empirical Pareto frontier. We can observe several similarities and differences between these two figures. First, the general shapes of the observed Pareto frontiers are the same. For large values of the partisan symmetry score, one can dramatically increase the achieveable compactness score by relaxing the demand for a high partisan symmetry score a little bit. We do not see a similar effect for large values of the compactness score, where the trade-off between compactness and partisan symmetry appears roughly linear everywhere except at the extreme end of the partisan symmetry score. In both states, we see that it is possible to find plans with nearly perfect partisan symmetry scores. We show the plan with the highest compactness score and highest partisan symmetry score in Figures 8 to 11 and the full set of Pareto-optimal plans are available online, along with replication code.[2]

While the shapes of the plots are similar, the numerical values of the scores associated to points at the Pareto frontier are very different in the two figures, which we highlight in Figure 6. In North

[2]https://zachschutzman.com/tradeoffs-fair-dist

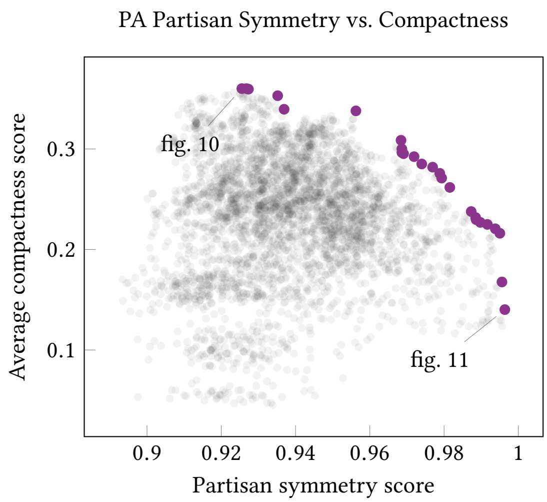

Figure 5: Comparison of partisan symmetry and compactness for Pennsylvania. Each point corresponds to one plan in the sample. Purple points are Pareto-optimal, grey points are Pareto-dominated.

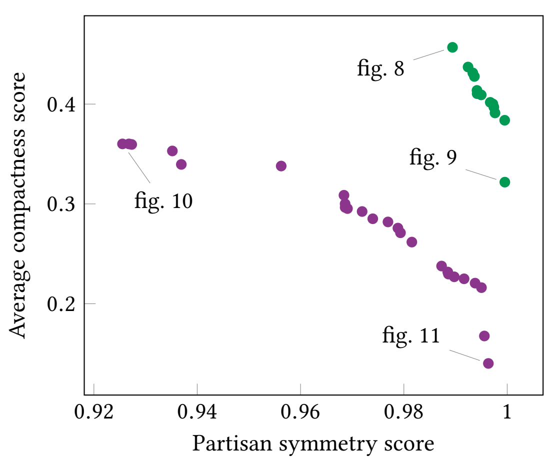

Figure 6: The Pareto-optimal points from Figures 4 and 5.

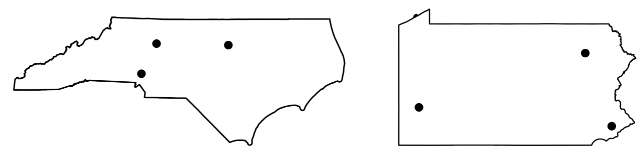

Figure 7: The approximate locations of urban areas in North Carolina and Pennsylvania

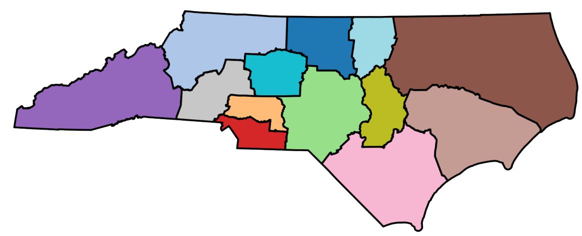

Figure 8: The most compact Pareto-optimal plan for North Carolina.

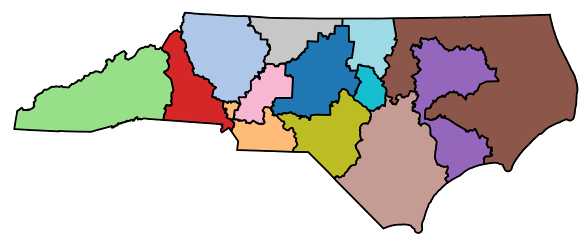

Figure 9: The Pareto-optimal plan with the highest degree of partisan symmetry for North Carolina.

Carolina, the most compact plans we found have a partisan symmetry of roughly 0.99, whereas in Pennsylvania, the most compact plans have a partisan symmetry score around 0.92. As a point of reference, the congressional districts enacted in North Carolina in 2016 and those enacted in Pennsylvania in 2011 were found in court to be egregious Republican-favoring gerrymanders and have partisan symmetry scores around 0.9 with respect to the election data used here, so a score of 0.92 suggests that this plan does indeed have a significant partisan tilt.

One explanation for this is the differing *political geography* of the two states [8]. Pennsylvania has high concentrations of Democrats in the densely populated corners of the state: the Philadelphia, Pittsburgh, and Scranton–Wilkes-Barre areas. On the other hand, the vast middle of the states has a much more sparse population and is largely Republican-favoring, although the Democratic tilt of the corners is much stronger than the Republican tilt of the middle. This means that, even though the balance of Democrats and Republicans is roughly equal, the urban districts will 'use up' more of the Democratic vote than the rural districts do of the Republican vote. Because the Democratic centers are geographically distant from each other, it is difficult to draw districts to balance this effect which are highly compact. On the other hand, North Carolina's population is much less concentrated. The largest county in North Carolina has about two-thirds the population of the largest county in Pennsylvania. Furthermore, there are a number of metropolitan regions with high concentrations of Democrats in the middle of the state, including the Raleigh–Durham and the Greensboro areas. The city of Charlotte is also somewhat centrally located in the state. For this reason, districts can remain highly compact and also include portions of these urban regions and rural regions, which helps to balance the asymmetry that arises from large numbers of Democrats living in more dense areas, as in Pennsylvania. The difference in achievable compactness scores may be attributable to the shapes of the precincts themselves, rather than any deeper political reason.

This analysis demonstrates that the 'cost' of partisan symmetry in terms of compactness (and vice versa) is different in the two states.

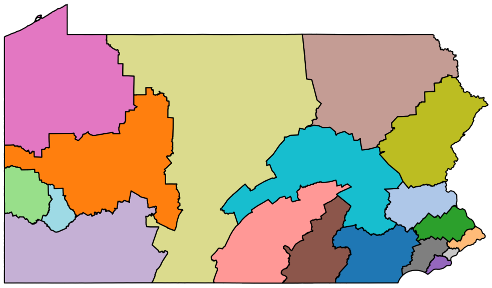

Figure 10: The most compact Pareto-optimal plan for Pennsylvania.

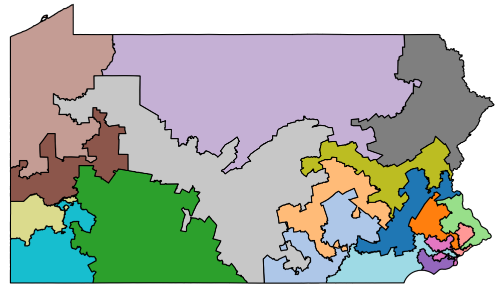

Figure 11: The Pareto-optimal plan with the highest degree of partisan symmetry for Pennsylvania.

In North Carolina, adhering to a neutral criterion of compactness gives us a high degree of partisan symmetry almost for free. We can see in Figure 9 that, with the exception of the two in the eastern portion of the state, the districts are relatively nicely shaped with much of the noncompactness coming from the jagged boundary, in contrast with the contorted shapes in Figure 11. This suggests that the converse is true as well: in North Carolina, aiming for districts which treat the two parties symmetrically doesn't require a severe deviation from nicely shaped districts.

On the other hand, in Pennsylvania, seeking a high degree of partisan symmetry comes at a high cost in terms of compactness. In Figure 11, we can see that in order to achieve partisan symmetry, the districts must contort around the Democratic strongholds to properly distribute votes among the less dense, Republican-leaning rural areas. In the southeast, we see five districts extending little tendrils into the Philadelphia area, in the southwest we see the Pittsburgh area divided among four districts. The large purple district across the northern part of the state balances a chunk of the Scranton–Wilkes-Barre area with a massive swath of low population rural regions along the New York border. Where there is a strongly Democratic district in the Raleigh–Durham area and a strongly Republican district in the northwestern part of the state and the remaining 11 districts balance mostly rural Republican populations with urban Democratic ones, but splitting up these urban areas does not require drawing the same kinds of contorted shapes as are necessary in Pennsylvania. In contrast, the districts in Figure 10 are much more regularly shaped, but achieve a very low degree of partisan symmetry. The four districts nestled in the southeast portion of the state as well as the teardrop shaped one in the southwest encompassing much of Pittsburgh are very strongly Democratic while most of the others have a solid, but relatively weaker, Republican tilt.

5 DISCUSSION AND FUTURE WORK

This work points toward several avenues for future research, and we highlight a few here. First, we only considered two instantiations of two particular notions of fairness in this domain. Repeating this analysis for other partisan measures, such as the competitiveness of districts, or other neutral procedures, such as avoiding the splitting of municipalities or counties, would shed more light on what the space of possible districting plans looks like. Additionally, we demonstrate our analysis on Pennsylvania and North Carolina, and the results are considerably different. We posit that this is due to the political geographies of the two states, and examining this effect is an important thread for understanding what kinds of reforms might or might not be effective in various jurisdictions. Future work could use more sophisticated mathematical and statistical techniques to describe a relationship between political geography and the trade-offs we consider here. Our analysis suggests that a one-size-fits-all approach to drawing 'fair' districts is inappropriate and that individual states and localities should carefully consider the relevant trade-offs when redistricting or implementing redistricting reform initiatives. One factor ignored in this analysis, which is critical to the process of drawing districts, is *respecting communities-of-interest*. Even defining and locating geographically such communities is a very difficult problem, let alone the determination of whether or not to preserve that group in a single district. We therefore propose our analysis as a framework for discussion about trade-offs in redistricting rather than as a policy recommendation.

In this work, we have demonstrated with a simple model that demanding districts be drawn to be as compact as possible and demanding that they satisfy a notion of partisan symmetry are incompatible, but to different degrees depending on the particular features of the geographic region in question. Since existing proposals and methodologies for automated and algorithmic redistricting involve finding an approximate solution to an optimization problem, it is important to understand how changing the objective function of these procedures can affect the outcome. As more jurisdictions consider redistricting reforms, they should be cautious about abdicating the line drawing process to algorithms which encode values different from those of the voters who use the districts to elect their representatives.

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